J. Fluid Mech. (1994), *vol.* 273, *pp.* 169–187

Modulation of the growth rate of short surface capillary–gravity wind waves by a long wave

By YU. I. TROITSKAYA

Institute of Applied Physics, Russian Academy of Sciences, Nizhny Novgorod, Russia

(Received 20 April 1992 and in revised form 22 February 1994)

Modulation of the growth rate of short capillary–gravity surface wind waves in the presence of a long wave with steepness much smaller than the maximum is studied theoretically. The Miles (1962) mechanism taking into account the viscous wave stresses in the air flow is considered to be the main process of short-wave generation. The short-wave growth rate is defined by the wind velocity gradient in the viscous sublayer of the logarithmic boundary layer. The long wave propagating on the wave surface induces an additional component of the wind velocity gradient oscillating with the length and time periods of the long wave, which results in modulation, with the same period, of the growth rate of the short wave riding on the long one. The growth-rate modulation amplitude depends on the parameter M being of the order of the relation between the oscillating and the mean wind velocity gradients in the viscous sublayer

$$M = \frac{2kac}{u_*^2}(ck\nu_a)^{1/2}$$

(where c, k, a are the phase velocity, the wavenumber and the elevation amplitude of the long wave; ν_a is the viscosity coefficient in the air; u_* is the wind friction velocity). When $M = O(1)$ (weak winds and long waves) the oscillating component of the short-wave growth rate is of the same order as the mean one. If M is much smaller than unity, then the relative amplitude of the growth rate is of the same order as the steepness of the long wave.

1. Introduction

Short waves of the centimetre bandwidth are known to form backscatter from the sea surface arising from Bragg scattering. So, when the theory of the imaging of long surface waves by radar is constructed, the problem of determining the variations of the short-surface-wave spectrum arises. Very large differences in the scales of short and long waves are typical for this problem: the respective wavelengths are 1–10 cm and 10–10^3 m; the time periods are 0.04–0.25 s and 2.5–25 s; the phase velocities are 25–40 cm s^{-1} and 4–40 m s^{-1}. Also, the steepness of long surface waves of 10–1000 m bandwidth is much smaller than the maximal value (0.142π) for typical ocean conditions.

Two mechanisms of interaction which result in modulation of short waves with the period of a long wave have already been considered. First, higher harmonics are generated near the crest of a steep gravity wave (Longuet-Higgins 1963 and Ruvinsky, Feldstein & Friedman 1991). But this mechanism proved to be effective only when the long wave is the almost-highest one. And the greater the wavelength of the long wave, the closer its steepness should be to the maximum value for the short waves to be generated effectively. Thus this mechanism is not necessarily applicable for long ocean waves with small steepness.

Another well-known mechanism is the transformation of a short wave on the variable flow of a long wave (Longuet-Higgins & Stewart 1960, 1961; Longuet-Higgins 1987; Phillips 1981; Shyu & Phillips 1990). But since there is no resonance between the phase velocity of the long wave and the group velocity of the short wave in the case under consideration, this mechanism provides small values of modulation of the short-wave amplitude (of the order of the long-wave steepness).

Another mechanism leading to the modulation of the short-wind-waves field was considered in the paper by Valenzuela & Wright (1979), where the modulation of the growth rate of short wind waves by the long waves was taken into account. In that paper this mechanism was considered only phenomenologically. Preliminary calculations of the modulation of the short wave by the long wave were done by Landahl, Widnall & Hultgren (1978), where the modified Orr–Sommerfeld equation for short-scale disturbances in the air obtained by the two-scale method was solved. But the calculations were carried out only for a limited set of parameters of the wind and waves, namely for a wind friction velocity $u_* = 30\ \mathrm{cm\ s^{-1}}$, wavelengths of the long waves of 100, 75, 36, 20 and 16.5 cm and wavelengths of the short waves of 2, 1, 0.75 and 0.6 cm. These calculations were for waves in laboratory channels. The main purpose of the present work is to calculate in detail the growth rate of short waves in the presence of long waves under conditions more typical for oceans.

The qualitative mechanism of modulation of the short-wave growth rate is discussed below. The Miles mechanism taking into account the viscous stress (Miles 1962) is considered as the main process of generation of the centimetre waves. In this case the value of the wave growth rate β is determined by two independent parameters, namely the gradient of the wind velocity U_{0z} in the viscous sublayer and the wavenumber κ of the short wave. It is well known that (Phillips 1977)

$$U_{0z} = u_*^2/\nu_a,$$

where u_* denotes the wind friction velocity, and ν_a denotes the viscosity coefficient of the air, i.e.

$$\beta = \beta(u_*^2/\nu_a, \kappa).$$

Consider a long surface wave with wavenumber k, phase velocity c and elevation amplitude a propagating in water. Then variable flow arises on the water surface. In the reference frame moving with the long-wave phase velocity, the horizontal flow at the surface is, to first order in the steepness (ka),

$$u = cka\cos kx - c. \tag{1.1}$$

And in the reference frame under consideration the phase velocity $\hat{c}_s$ of the short wave is

$$\hat{c}_s = -c + c_s + cka\cos kx. \tag{1.2}$$

Here c_s is the local intrinsic phase velocity of the short waves. It was shown by Longuet-Higgins & Stewart (1960, 1961) that the wavenumber κ, the phase velocity c_s and the elevation amplitude of the short gravity wave appeared to be modulated:

$$\left.\begin{aligned} \kappa &= \kappa^{(0)}(1+ka\cos kx), \\ c_s &= c_s^{(0)}(1-ka\cos kx), \\ n &= n^{(0)}(1+ka\cos kx). \end{aligned}\right\} \tag{1.3}$$

If there is a long wave on the water surface, it induces a disturbance in the air, which has the same length and time periods as the wave in the water, so modulation of the short-wave growth rate arises. This disturbance is considered next to demonstrate why the growth rate of the short waves can be strongly modulated.

If there were no viscosity in the air or in the water, then the disturbances of the vertical velocity would be continuous at the air–water boundary, but the disturbances of the horizontal velocity would have a discontinuity of order akc. But if the viscosity of the air and water are taken into account, then the no-slip condition is valid on their boundary, i.e. the horizontal velocity is continuous as well. In this case a wave boundary layer with thickness $\delta_w \ll 1/k$ appears in the air, i.e. flow with a large oscillating gradient of the air flow velocity (of order cka/δ_w) arises near the water surface. It should be emphasized that the oscillating part of the wind velocity gradient determines the oscillating part of the growth rate of the short waves. Indeed, the air velocity profile in the reference frame moving with the long-wave phase velocity is

$$u = U_0(z) - c + cka[\cos kx + f(z/\delta_w)\cos(kx-\phi)]. \tag{1.4}$$

$f(z/\delta_w)$ is the velocity profile in the long-wave boundary layer, and according to the no-slip conditions $f|_{z=0} = 0$. The air flow velocity on the air–water boundary is defined by (1.1). But the wave–flow interaction is known to be determined by their relative velocity. More exactly, the difference $u - \hat{c}_s$ in the Orr–Sommerfeld equation determines the dispersion properties of the short waves:

$$u - \hat{c}_s = -c_s + U_0(z) + ckaf(z/\delta_w)\cos(kx-\phi). \tag{1.5}$$

Comparing (1.4) and (1.2) shows that the oscillating part of the phase velocity of the short waves ($cka\cos kx$) is exactly equal to the oscillating air flow velocity on the water surface (due to the no-slip condition). As a result there is no oscillating term ($cka\cos kx$) in (1.5). If the scale of the waves is small enough that the region of intensive wave–flow interaction is in the viscous sublayer, then their growth rate is determined by the gradient of $u - \hat{c}_s$:

$$u_z = \frac{u_*^2}{\nu_a} + \frac{cka}{\delta_w} f\left(\frac{z}{\delta_w}\right)\cos(kx-\phi), \tag{1.6}$$

and it follows from (1.6) that strong modulation of the air flow velocity gradient (and strong modulation of the short surface waves depending on it) should be expected for weak winds and large wavelengths of long waves.

The growth rate is calculated in the same way as in Miles (1962). Namely, taking into account that the ratios of the densities ($\rho_a/\rho_w \sim 10^{-3}$) and the dynamical viscosity coefficients ($\rho_a \nu_a/\rho_w \nu_w \sim 10^{-2}$) of the water and the air are small enables the problem to be solved in the following way. First the hydrodynamical problem of surface-wave generation by normal and tangential air stresses forcing the water surface is solved. Then the aerodynamical problem of the air flow disturbance induced by the surface waves is considered. The normal and tangential stresses forcing the water surface are calculated. Thus the closed dispersion relation for the waves in the air–water flow is obtained. When the modulation of the growth rate of short waves by a long wave is considered, each of the two problems becomes split into two further ones: for a long wave and for a short wave on a long wave. Thus four problems arise: first, a long surface wave in water; second, a short wave in water on a long wave; third, a long-wave disturbance in air; and, fourth, a short wave on a long wave in the air.

The main difficulty of this task is caused by the thickness of the viscous sublayer in the air, δ, being much smaller than the elevation amplitude of long surface waves in typical ocean conditions, i.e. the wave disturbance in the air being strongly nonlinear. Fortunately it is simplified by the fact that the flow around a small-steepness wave is without separation. And it is pointed out in the work by Benjamin (1959), referring to Schlichting (1955), that the distortion of the velocity profile near the wavy surface is formed by bending of the streamlines of the undisturbed air flow to the first order in the surface curvature. If the problem is formulated in curvilinear two-dimensional coordinates with the wavy water surface being a coordinate line, the disturbances appear to be displacements from the bent streamlines of no more than the first order in ka (Benjamin 1959). The curvilinear coordinates proposed in Benjamin (1959), in which the water surface resulting from the long harmonic wave is a coordinate line to the first order of approximation in (ka), are used below for solution of the problem.

A point concerning the order of expansion (ka) being taken into account should be made here. There are several small parameters in the problem under consideration, namely ka, c_s/c, k/κ, and v_0/c, where v_0 denotes the wind drift flow velocity. They are independent, but of the same order. And since the dependence of c_s on ka (i.e. the quantity of order $c_s ka$) is taken into account, then the quantities of order $c(ka)^2$ (which are the same order as $c_s ka$) should be taken into account as well. However, they cause additional terms in the phase velocity much smaller than $O(c(ka)^2)$, which can be omitted (see the Appendix).

The structure of the paper is the following. In §2 the formulation of the problem in curvilinear coordinates is presented. In §§3–6 the hydrodynamical and aerodynamical problems for the long and the short waves are considered. In §7 the results of the numerical calculations of the modulation of the growth rate of the short waves on the long waves are presented and discussed. In §8 the influence of the modulation of the growth rate on the modulation of the spectral component of short wind waves with a long wave is discussed.

2. Formulation of the problem; curvilinear coordinates

Consider plane air shear flow over the water surface. Let two wave disturbances propagate in this two-layered system: a long wave with wavenumber k and elevation amplitude a and a short wave with wavenumber k_s and elevation amplitude a_s with $k \ll k_s$ and $a_s \ll a$. The steepness of both the long and the short waves is considered to be small, so that the values $(ka)^2$ and $(k_s a_s)^2$ can be omitted. Air flow over the water surface without separation is considered.

To consider large gradients of velocity in the viscous sublayer at the air–water interface one should use curvilinear coordinates in which the water surface bent by the long wave is a coordinate line at the first order of approximation in ka. Since the wave field should decrease with the distance from the boundary, different curvilinear coordinates are used in the water and in the air. Namely, in the water

$$s = x + \mathrm{i}a\,\mathrm{e}^{k(y+\mathrm{i}x)}, \quad \gamma = y - a\,\mathrm{e}^{k(y+\mathrm{i}x)}, \tag{2.1}$$

and in the air, according to Benjamin (1959)

$$\xi = x - \mathrm{i}a\,\mathrm{e}^{-k(y-\mathrm{i}x)}, \quad \eta = y - a\,\mathrm{e}^{-k(y-\mathrm{i}x)}. \tag{2.2}$$

The real parts of the right-hand sides of (2.1) and (2.2) are to be taken. This also applies to the complex expressions of this kind given below. It follows from (2.1) and (2.2) that surfaces $\eta = 0$ and $\gamma = 0$ coincide with the water surface disturbed by the long wave at the first order of approximation in ka. The Jacobians of transformations in the air and in the water at the first order of approximation in ka are respectively the following:

$$\mathscr{J}^w = 1 - 2ka\,\mathrm{e}^{k(\gamma+\mathrm{i}s)}, \tag{2.3}$$

$$\mathscr{J}^a = 1 + 2ka\,\mathrm{e}^{-k(\eta-\mathrm{i}\xi)}. \tag{2.4}$$

All the expressions below are presented in the curvilinear coordinates in the air (2.2). The expressions for the fields in the water are the same. The indices (*w*) and (*a*) for values in air and in water respectively will be given only if necessary.

The expression for the velocity $\boldsymbol{v}$ using the stream function ψ in the curvilinear coordinates will be required below. It is

$$\boldsymbol{v} = J^{1/2}\left[\boldsymbol{\xi}_0 \frac{\partial \psi}{\partial \eta} - \boldsymbol{\eta}_0 \frac{\partial \psi}{\partial \xi}\right]. \tag{2.5}$$

And the two-dimensional hydrodynamics equations in curvilinear coordinates are

$$\frac{\partial}{\partial \eta}(\psi_t) + \tfrac{1}{2}J_\xi(\psi_\xi^2 + \psi_\eta^2) + J(\psi_\eta \psi_{\xi\eta} - \psi_\xi \xi_{\eta\eta}) + \frac{1}{\rho}p_\xi + g y_\xi = \nu \frac{\partial}{\partial \eta}[J(\psi_{\xi\xi} + \psi_{\eta\eta})], \tag{2.6a}$$

$$-\frac{\partial}{\partial \xi}(\psi_t) + \tfrac{1}{2}J_\eta(\psi_\xi^2 + \psi_\eta^2) + J(\psi_\xi \psi_{\xi\eta} - \psi_\eta \psi_{\xi\xi}) + \frac{1}{\rho}p_\eta + g y_\eta = -\nu \frac{\partial}{\partial \xi}[J(\psi_{\xi\xi} + \psi_{\eta\eta})], \tag{2.6b}$$

denoting the density ρ and the molecular viscosity ν. Eliminating p from (2.6*a, b*) gives the hydrodynamical system in terms of vorticity and stream function, which has the following form in the curvilinear coordinates:

$$\Xi_t + J(\psi_\eta \Xi_\xi - \psi_\xi \Xi_\eta) = \nu J(\Xi_{\xi\xi} + \Xi_{\eta\eta}), \tag{2.7a}$$

$$\Xi = J(\psi_{\xi\xi} + \psi_{\eta\eta}). \tag{2.7b}$$

Here Ξ is the vorticity of the flow.

The system (2.7) appears to be more convenient for numerical calculations than (2.6), and it is used below to consider wave disturbances in the air where the numerical method can be employed.

The systems (2.6) and (2.7) should be completed by the kinematical and dynamical boundary conditions on the air–water surface, which must be written at the air–water boundary surface disturbed by both the long wave and the short wave:

$$\frac{\partial H}{\partial t} + J\left(\frac{\partial H}{\partial \xi}\frac{\partial \psi}{\partial \eta} - \frac{\partial \psi}{\partial \xi}\frac{\partial H}{\partial \eta}\right)\bigg|_{H=0} = 0. \tag{2.8}$$

Here

$$H = N[x(\xi, \eta)] + n^s(x(\xi, \eta), t) - y(\xi, \eta), \tag{2.9}$$

N and n^s denoting the elevations of the long and short waves respectively.

The no-slip condition is equivalent to the continuity of the ξth component of the velocity at the water–air boundary. Taking into account (2.5) yields

$$(J^w)^{1/2}\frac{\partial \psi^w}{\partial \gamma} = (J^a)^{1/2}\frac{\partial \psi^a}{\partial \eta}. \tag{2.10}$$

The dynamical boundary conditions are the continuity of the normal and tangential stresses on the water–air boundary surface. They are

$$p^a + \rho_a \nu_a \sigma_n^a|_{\eta=n} = p^w + \rho_w \nu_w \sigma_n^w + \frac{T\rho_w}{R}\bigg|_{\gamma=n},$$

$$\rho_a \nu_a \sigma_\tau^a|_{\eta=n} = \rho_w \nu_w \sigma_\tau^w|_{\gamma=n}.$$

Here $T\rho_w/R$ is the surface tension force: R denotes the radius of curvature of the wavy surface; p^a, p^w are the pressures in the air and in the water following from system (2.6);

σ_n^a, σ_n^w, σ_τ^a, σ_τ^w are the viscous normal and tangential stresses in the water and in the air respectively. They are expressed as follows:

$$\sigma_n = \frac{2(\xi_x^2-\xi_y^2)}{J^a}(\psi_{\xi\eta}(\xi_x^2-\xi_y^2)+\psi_{\xi\xi}\xi_x\xi_y+\psi_{\eta\eta}\eta_x\eta_y+\psi_\xi\xi_{xy}+\psi_\eta\eta_{xy}), \tag{2.11}$$

$$\sigma_\tau = ((\xi_x^2-\xi_y^2)(\psi_{\eta\eta}-\psi_{\xi\xi})+4\psi_{\xi\eta}\xi_x\xi_y-2\psi_\xi\eta_{xy}+\psi_\eta\xi_{xy}). \tag{2.12}$$

We will now consider the four problems outlined in the Introduction.

3. A long wave in water

Suppose that a wind drift current with velocity profile $v_0(\gamma)$ is present in the water. In this case the stream function of the flow disturbed by the long wave is

$$\psi^w = \int^\gamma (v_0(\gamma_1)-c)\,\mathrm{d}\gamma_1 + a\psi^l(s,\gamma). \tag{3.1}$$

Consideration of the long-wave disturbance in the water follows the paper by Miles (1962), but the wind drift flow in the water is taken into account here, and the analytical expressions for the dispersion relation and the stream function are obtained for the typical situation of the drift flow velocity v_0 being small in comparison with the phase velocity of the long wave c. It differs from the paper by Valenzuela (1976) where the influence of the drift flow was considered exactly in the numerical model.

The disturbance of the stream function is presented the following way (as in Benjamin 1959):

$$\psi^l = [\phi^l(\gamma)+(v_0-c)\,\mathrm{e}^{k\gamma}]\,\mathrm{e}^{\mathrm{i}ks}. \tag{3.2}$$

Long waves with wavelengths of the order 10 m or more will be considered. Neglecting the effect of molecular viscosity on the long waves enables the Rayleigh equation for ϕ^l to be obtained from the system (2.6) at the first order of approximation in ka:

$$(v_0-c)(\phi^l_{\gamma\gamma}-k^2\phi^l)-v_{0\gamma\gamma}\phi^l = 0. \tag{3.3}$$

It follows from the kinematical boundary condition (2.8) that to the first order in ka

$$\phi^l(\gamma=0)+v_0(0)-c = 0. \tag{3.4}$$

The normal and tangential air stresses are known to be small for these long waves (see Phillips 1977). Taking this into account and omitting the surface tension for the long wave yields the dynamical boundary condition

$$-g+\phi^l v_{0\gamma}-(v_0-c)\,\phi^l_\gamma = 0. \tag{3.5}$$

The phase velocities c of the waves are of order 10 m s^{-1}. Winds weaker than that, for which waves of the bandwidth considered are the peak ones, are considered below, which means that the wind velocity is less than 10 m s^{-1} and the drift flow velocity is less than $10(\rho_a/\rho_w)^{1/2}$ m s^{-1}, or less than 0.3 m s^{-1}. Thus $v_0 \ll c$.

Solving equation (3.3) with the boundary conditions (3.4) and (3.5) yields the dispersion relation for the surface water waves. Taking into account that v_0/c is small the solution of (3.3) and the dispersion relation should be sought as a series in v_0/c. At the first order of approximation in (v_0/c) the dispersion relation is

$$c = \left(\frac{g}{k}\right)^{1/2}\left(1+2k\int_{-\infty}^0 \frac{v_0(\gamma)}{c}\,\mathrm{e}^{2k\gamma}\,\mathrm{d}\gamma\right). \tag{3.6}$$

If the scale of the function $v_0(\gamma)$ is much smaller than the wavelength of the disturbance $1/k$, then the expression for the disturbance of the stream function $a\psi^l$ is

$$a\psi^l = 2ka\,e^{k\gamma}\int_0^\gamma v_0(\gamma)\,d\gamma. \tag{3.7}$$

It follows from (3.7) that in the curvilinear coordinates considered here the long-wave disturbances of the stream function are of order $av_0 \ll ac$ (ac is the order of the stream function disturbance in Cartesian coordinates). The disturbances of the stream function being small means that the streamlines are close to the coordinate lines in the curvilinear coordinates, and the coordinate lines of the transformation (2.1) coincide with the streamlines of the surface wave on deep water moving with velocity $-c$.

Although the disturbances of the stream function are small, the velocity disturbances are essential. Indeed, the expression for the horizontal velocity is

$$v_{hor} = (J^w)^{1/2}\frac{\partial\psi^w}{\partial s} = v_0(\gamma) - c + ka\,e^{iks}(c+v_0(\gamma)), \tag{3.8}$$

where (2.5) is taken into consideration and values of order kL are omitted.

4. A short-wave disturbance on a long wave in water

In this section the dispersion relation for short waves modulated by a long wave is obtained. The procedure for obtaining of the dispersion relation is similar to that used in Miles (1962). Some peculiarities of the two scales should be taken into account.

In the presence of the short-wave and the long-wave disturbances the stream function is

$$\psi^w = \int(v_0 - c)\,d\gamma + 2ka\,e^{iks}\,e^{k\gamma}\int_0^\gamma v_0(\gamma)\,d\gamma + \psi^s, \tag{4.1}$$

ϕ^s being the short-wave disturbance of the stream function. Dimensionless variables are introduced:

$$\sigma = s/L; \quad \theta = \gamma/L; \quad \tau = tc/L,$$

where L is the scale of the velocity profile in the viscous sublayer in the air and c is the phase velocity of the long wave. The characteristic scale of the short wave $1/k_s$ is generally of order L (more exactly L is less than $1/k_s$) and $\mu = kL$ is a small parameter of order of or less than $k/k_s \ll 1$.

The problem of short–long wave interaction can be treated by the two-scale method considering 'fast' (τ, σ, θ) and 'slow' scales

$$T = \mu\tau; \quad \Sigma = \mu\sigma; \quad Q = \mu\theta.$$

In this case an expression for ψ^s should be sought in the following form:

$$\psi^s = \phi(\theta, Q, \Sigma, T)\,e^{-i\Omega\tau + i\Phi(\sigma, Q, \Sigma, T)} + \mu\phi_1^s.$$

Here Φ is the eikonal. By definition the dimensionless horizontal wavenumber of the short wave $\kappa = k_s L$ is the horizontal component of the eikonal Φ gradient. And in the curvilinear coordinates (2.1)

$$\kappa = (J^w)^{1/2}\frac{\partial\Phi}{\partial\sigma},$$

where J^w is defined in (2.3).

Suppose (similar to Miles 1962) that ϕ can be presented as the sum of the viscous and non-viscous terms:

$$\phi = A\Psi(\theta, Q, \Sigma, T) + B\omega(\theta, Q, \Sigma, T).$$

At zeroth order in μ the non-viscous term ψ appears to be a function of θ depending parametrically on Σ and obeying the Rayleigh equation

$$[(1+\alpha)(v_0(\theta)-v_0(0))-c_s](\Psi_{\theta\theta}-\kappa^2(1+2\alpha)\Psi)-v_{0\theta\theta}(1+\alpha\Psi) = 0. \tag{4.2}$$

Here
$$\frac{c_s}{c} = \left(\frac{\Omega}{\kappa}-\frac{v_0(0)}{c}(1+\alpha)+(1-\alpha)\right), \tag{4.3}$$

$$\alpha = ka\,\mathrm{e}^{\mathrm{i}\Sigma}.$$

Taking (3.8) into consideration easily shows that c_s is the local intrinsic phase velocity of the short waves. It should be mentioned that the difference $c_s - v_{hor}$ (or the relative velocity of the water and the phase velocity of the short waves) occurs in equation (4.2).

The viscous term is

$$\omega = \mathrm{e}^{m\theta}, \quad m = (1-\mathrm{i})\left[\frac{c_s L\kappa}{2\nu_w}(1+\alpha)\right]^{1/2}. \tag{4.4}$$

c_s can be determined as a solution of the dispersion equation for the short waves on the long wave. It is

$$\frac{\Psi_\theta}{\Psi}\left(1-\frac{c_0^2}{c_s^2}\right) = -\frac{v_{0\theta}(1-\alpha)}{c_s}+\frac{\nu_w}{\mathrm{i}\kappa c_s}\mathbb{D}+\frac{\rho_a}{\rho_w}\mathbb{G}. \tag{4.5}$$

Here
$$c_0^2 = \left(g+\frac{T}{L^2}\kappa^2(1+\alpha)\right)L\frac{\Psi}{\Psi_\theta}, \tag{4.6}$$

and $\mathbb{D}$, $\mathbb{G}$ denote the terms which describe dissipation of the short waves by viscosity and generation by wind respectively:

$$\mathbb{D} = \left\{-\left(\frac{\Psi_{\theta\theta\theta}}{\Psi}-3\kappa^2(1+2\alpha)\frac{\Psi_\theta}{\Psi}\right)+\frac{c_0^2}{c_s^2}\frac{\Psi_\theta}{\Psi}\left(\left(\frac{\Psi_{\theta\theta}}{\Psi}+\kappa^2(1+2\alpha)\right)(1-2\alpha)-4\kappa\alpha\frac{\Psi_\theta}{\Psi}\right)\right\}\Bigg|_{\theta=0},$$

$$\mathbb{G} = \frac{L}{\Psi c_s}\left[\mathscr{P}+\mathscr{T}\nu_a\,\mathrm{i}\left(\frac{c_0^2}{c_s^2}\frac{\Psi_\theta}{\Psi\kappa(1+a)}+2\alpha\right)\right](1+\alpha)\Bigg|_{\theta=0}.$$

Here $\mathbb{G}$ is determined by the normal ($\mathscr{P}$) and tangential ($\mathscr{T}$) stresses induced by the short wave in the air flow. To find them the problem of disturbances in the air flow over the water surface should be solved.

5. A long-wave disturbance in the air flow

The system of hydrodynamical equations for a long-wave disturbance in the air flow in curvilinear coordinates (2.2) in terms of stream function and vorticity follows from the system (2.7*a, b*). Taking into consideration the expressions for ξ, η and the Jacobian $\mathbb{J}^a$ to the first order of approximation in ka gives

$$\psi_\eta \Xi_\xi - \psi_\xi \Xi_\eta = \nu(\Xi_{\xi\xi}+\Xi_{\eta\eta}), \tag{5.1a}$$

$$\Xi = (1+2ka\,\mathrm{e}^{-k(\eta-\mathrm{i}\xi)})(\psi_{\xi\xi}+\psi_{\eta\eta}). \tag{5.1b}$$

ψ can be expressed as a sum of two terms, one of which depends on ξ and the other does not:

$$\psi = \int_0^{\eta} (u_0(\eta_1) - c)\, \mathrm{d}\eta_1 + \psi_1(\xi, \eta), \tag{5.2a}$$

$$\Xi = u_{0_\eta}(\eta) + \Xi_1(\xi, \eta). \tag{5.2b}$$

The kinematical boundary condition which is the consequence of (2.8) at the first order in ka is

$$\psi_{1\xi}|_{\eta=0} = 0. \tag{5.3}$$

At the first order in ka the no-slip condition (2.10) can be written the following way:

$$(1 + ka\, \mathrm{e}^{\mathrm{i}k\xi})\, \psi^a_\eta|_{\eta=0} = (1 - ka\, \mathrm{e}^{\mathrm{i}k\xi})\, \psi^w_\gamma|_{\gamma=0}.$$

Taking into consideration that ψ^w is defined by (3.1), (3.7) and that

$$v_0(0) = u_0(0)$$

yields the boundary condition for ψ_1

$$\psi_{1\eta}|_{\eta=0} = 2cka\, \mathrm{e}^{\mathrm{i}k\xi}, \tag{5.4}$$

and

$$\psi_\eta|_{\eta=0} = -c + u_0(0) + 2cka\, \mathrm{e}^{\mathrm{i}k\xi}. \tag{5.5}$$

Long–short wave interaction is considered below. This interaction takes place mainly in the viscous sublayer. It follows from (5.5) that velocity disturbance is much smaller than the mean velocity (of order ka). One can also see that this relation remains valid on the scale of the short wave. But the disturbance of the velocity gradient (the vorticity) may be of the same order as the vorticity of the mean flow. This case will be called the nonlinear regime. But in this case the vorticity disturbances prove to obey the linearized form of the hydrodynamical equations, since their horizontal scale is much larger than the vertical one.

Consider here some estimates. First, the condition of the values of the vorticity of the wave field Γ_w being of the same order as the mean values of the vorticity Γ_0 is formulated below. Γ_w is obviously of order akc/δ_w, where δ_w is the scale of the long-wave viscous sublayer, and $\Gamma_0 = u_*^2/\nu_a$ (Phillips 1977). The condition $\Gamma_w = \Gamma_0$ defines the boundary between the linear and nonlinear regimes. One can easily estimate the scale of the wave field δ_w for the nonlinear regime. In this case the variation of the mean wind velocity on the scale of the wave field is of order $\Gamma_0 \delta_w \sim \Gamma_w \delta_w \sim kac$, which is much smaller than c. In this case the scale of the wave field close to the water–air boundary is of the order of the scale of the oscillating boundary layer (see, for example, Lighthill 1978), namely

$$\delta_w = (\nu_a/kc)^{1/2}. \tag{5.6}$$

Taking that into consideration yields the following expression for the boundary between the linear and nonlinear regions:

$$u_*^2/\nu_a = (kc/\nu_a)^{1/2}\, akc. \tag{5.7}$$

It should be mentioned that the nonlinear regime of the long-wave disturbance in the air corresponds to the swell, i.e. to a wave with phase velocity much greater than the velocity of the peak wind waves equal to $20u_*$. Indeed, it follows from (5.7), that

$$20u_*/c = 20(kak\delta_w)^{1/2} \ll 1.$$

Expressions for the wave fields of the velocity and normal and tangential stresses for the nonlinear regime can be easily obtained. In this case the following estimates for the stream function and its derivatives are valid. They follow from expression (5.2) for ψ and the boundary conditions (5.3) and (5.5):

$$\begin{aligned}\psi &\sim O(c\delta)+O(cka\delta_w)\,\mathrm{e}^{\mathrm{i}k\xi},\\ \psi_\eta &\sim O(c+u_0)+O(cka)\,\mathrm{e}^{\mathrm{i}k\xi},\\ \psi_\xi &\sim O(ckak\delta_w)\,\mathrm{e}^{\mathrm{i}k\xi},\\ \psi_{\xi\eta} &\sim O(ck^2a)\,\mathrm{e}^{\mathrm{i}k\xi},\\ \psi_{\xi\xi} &\sim O(ckak^2\delta_w)\,\mathrm{e}^{\mathrm{i}k\xi},\\ \Xi \sim \psi_{\eta\eta} &\sim O(u_0/\delta)+O(cka/\delta_w)\,\mathrm{e}^{\mathrm{i}k\xi},\end{aligned}$$

and so on. Here δ_w is defined by (5.6) and

$$\delta = 10\nu_a/u_* \tag{5.8}$$

is the scale of the viscous sublayer.

Taking this estimate into consideration and omitting small terms of order $k\delta$, $k\delta_w$, and $(ka)^2$ gives the following system from (5.1 *a*, *b*):

$$-c\Xi_{1\xi} = \nu_a\Xi_{1\eta\eta}, \quad \Xi_1 = \psi_{1\eta\eta}. \tag{5.9\textit{a, b}}$$

The solution of system (5.9) obeying the boundary conditions (5.3), (5.5) and limited at infinite distance from the air–water boundary is

$$\Xi_1 = 2ckar\,\mathrm{e}^{-r\eta}, \quad \psi_1 = \frac{2cka}{r}(1-\mathrm{e}^{-r\eta}), \tag{5.10\textit{a, b}}$$

where $r = \mathrm{e}^{-3\pi\mathrm{i}/4}/\delta_w$, and δ_w is defined in (5.6).

Now consider the linear regime of a wave in air, when the vorticity disturbances are much smaller than the mean value, i.e. $\Gamma_0 \gg \Gamma_w$. In this case the linear approximation is valid and ψ_1 obeys the following linear equation:

$$(u_0-c)\,\Xi_1 - u_{0\eta\eta}\,\psi_1 = \frac{\nu_a}{\mathrm{i}k}(\Xi_{1\eta\eta}-k^2\Xi_1), \tag{5.11\textit{a}}$$

$$\Xi_1 = (\psi_{1\eta\eta}-k^2\psi_1)+2kau_{0\eta}\,\mathrm{e}^{-k\eta}. \tag{5.11\textit{b}}$$

It should be mentioned that system (5.11 a, b) transforms into (5.9 *a*, *b*) when $c \gg u_0$ and the characteristic vertical scale of the wave disturbance is much smaller than $1/k$. Also the set of wind and long-wave parameters considered by Landahl *et al.* (1978) (see the Introduction) correspond to the linear regime of the long-wave disturbance.

To conclude this section the expression for the gradient of the air flow velocity Γ near the water surface is discussed. It follows from (5.2) that

$$\Gamma = (1+ka\,\mathrm{e}^{\mathrm{i}k\xi})\,u_{0\eta}(\eta)+\psi_{1\eta\eta}\,\mathrm{e}^{\mathrm{i}k\xi}.$$

The oscillating component of the wind velocity is the sum of two terms. The curvature of the water surface is responsible for the first term, and the second term, proportional to $\psi_{1\eta\eta}$, arises due to the oscillations of the water surface. The first term is of the first order in ka, relating to the mean gradient, and its phase coincides with the phase of the long wave. It obviously follows from (5.10 *b*), (5.6) and (5.7) that the second term is of the same order as the mean gradient $u_{0\eta}$ for the nonlinear regime. And it is of first order in ka for the linear regime. For the nonlinear regime the phase of the oscillating part of the wind velocity gradient is determined by the phase of $\psi_{1\eta\eta}$.

Differentiating (5.10b) easily gives that the phase of Γ should be close to $(-3\pi/4)$. For the linear regime the phase of the oscillating part of the gradient is determined by both of its components.

The expression for Γ can be easily obtained for the case $kau_{0\eta} \sim \psi_1$, Ξ_1, but $u_0 \ll c$. Then the linear system (5.11a, b) is valid. It then follows from (5.11b), that

$$\psi_{1\eta\eta} = \Xi_1 - 2ka\,e^{ik\xi}\,u_{0\eta}(\eta).$$

Seeking the solution to (5.11) as a series in (u_0/c) gives that Ξ_1 is determined by (5.10a). Then

$$\Gamma = (1 - k\alpha\,e^{ik\xi})\,u_{0\eta}(\eta) + 2ckar\,e^{-r\eta}\,e^{ik\xi}.$$

The phase of the oscillating part of the velocity gradient Γ obviously tends to $-\pi$ with growth of the wind velocity.

6. A short-wave disturbance on a long wave in air

The stream function and the vorticity for this case can be expressed as a sum of (5.2a,b) and the short-wave disturbances ψ^s and Ξ^s:

$$\psi = \int_0^\eta (u_0(\eta_1) - c)\,d\eta_1 + \psi_1(\eta)\,e^{ik\xi} + \psi^s(\xi, \eta), \tag{6.1a}$$

$$\Xi = u_{0\eta}(\eta) + \Xi_1(\eta)\,e^{ik\xi} + \Xi^s(\xi, \eta). \tag{6.1b}$$

Here ψ_1, Ξ_1 are the solutions to (5.9) or (5.11) and $\psi^s(\xi, \eta)\,\Xi^s(\xi, \eta)$ are functions with characteristic horizontal scale of ξ much less than $1/k$.

The problem will be formulated in dimensionless coordinates like (4.1):

$$\zeta = \xi/L; \quad h = \eta/L; \quad \tau = tc/L,$$

where L is the vertical scale of the wind velocity defined by (5.8); $\mu = kL$ is the small parameter.

Considering the short-wave disturbance in the air we make use of the two-scales method as in §4, i.e. the short-wave disturbances of the stream function ψ^s, the vorticity Ξ^s, the pressure p^s and the short-wave elevation n^s should be sought in the following form:

$$\psi^s = \psi^s_0(\tau, \zeta, h, T, Z, \Theta) + \mu\phi^s_1,$$
$$\Xi^s = \Xi^s_0(\tau, \zeta, h, T, Z, \Theta) + \mu\Xi^s_1,$$
$$p^s = p^s_0(\tau, \zeta, h, T, Z, \Theta) + \mu p^s_1,$$
$$n^s = n^s_0(\tau, \zeta, T, Z) + \mu n^s_1,$$

denoting $\Theta = \mu h$, $Z = \mu\zeta$. (It should be mentioned here that the difference between Z and $\Sigma = \mu s$ is of order $(ka)^2$.)

At the zeroth order in μ and at the first order in ka the following system of equations for the lowest terms of the short-wave disturbances (ψ^s_0, Ξ^s_0) can be obtained from (2.7a,b):

$$\frac{\partial \Xi^s}{\partial \tau} + (1 + 2ka\,e^{\Theta + iZ})\left[\left(\frac{u_0 - c}{c} + \frac{\psi_{1h}}{Lc}\right)\Xi^s_\zeta - \left(\frac{u_{0hh}}{cL^2} + \frac{\Xi_{1hh}}{Lc}\right)\psi^s_\zeta\right]$$
$$= \frac{\nu_a}{Lc}(1 + 2ka\,e^{\Theta + iZ})(\Xi^s_{\zeta\zeta} + \Xi^s_{hh}), \tag{6.2a}$$

$$L^2 \Xi^s = (1 + 2ka\,e^{\Theta + iZ})(\psi^s_{\zeta\zeta} + \psi^s_{hh}). \tag{6.2b}$$

The equation for the short-wave disturbances, which follows from (2.6*a*), will be necessary for calculating the normal stress on the water surface. It is

$$\frac{\partial}{\partial h}(\psi^s_\tau)+(1+2ka\,\mathrm{e}^{\Theta+\mathrm{i}Z})\left[\left(\frac{u_0-c}{c}+\frac{\psi_{1h}}{Lc}\right)\psi^s_{h\zeta}-\left(\frac{u_{0h}}{c}+\frac{\psi_{1hh}}{Lc}\right)\psi^s_\zeta\right]$$
$$+\frac{L}{c}\frac{p^s_\zeta}{\rho_a}=\frac{\nu_a}{Lc}\frac{\partial}{\partial h}((1+2kca\,\mathrm{e}^{\Theta+\mathrm{i}Z})(\psi^s_{\zeta\zeta}+\psi^s_{hh})). \quad (6.2c)$$

It follows from (6.2*a, b*) that at the zeroth order in μ the solution can be sought in the following form:†

$$\psi^s=\chi(h,Z)\,\mathrm{e}^{-\mathrm{i}\Omega\tau+\mathrm{i}\Phi(\zeta,Z)},$$
$$\Xi^s=\Xi(h,Z)\,\mathrm{e}^{-\mathrm{i}\Omega\tau+\mathrm{i}\Phi(\zeta,Z)}.$$

Here $\Phi(\zeta,Z)$ is the eikonal of the short waves. In air the eikonal and the horizontal wavenumber κ are connected in the following way:

$$\kappa=(J^\alpha)^{1/2}\frac{\partial\Phi}{\partial\zeta}. \quad (6.3)$$

Taking (6.3) into account gives a system of equations for the complex amplitudes of the short-wave disturbances of the stream function $\chi(h,Z)$ and vorticity $\Xi(h,Z)$:

$$(U(h,Z)-c_s)\,\Xi-X_h(h,Z)\,\chi=\frac{\nu_a}{\mathrm{i}\kappa}(1+2\alpha)\,[\Xi_{hh}-\kappa^2(1-2\alpha)\,\Xi], \quad (6.4a)$$

$$\Xi=(1+2\alpha)\,[\chi_{hh}-\kappa^2(1-2\alpha)\,\chi], \quad (6.4b)$$

where
$$U(h,Z)=(u_0(h)-u_0(0))\,(1+\alpha)+\left(\frac{\psi_{1h}}{L}-2kac\right)\mathrm{e}^{\mathrm{i}\Sigma}, \quad (6.5a)$$

$$X(h,Z)=u_{0h}(h)\,(1+\alpha)+\Xi_1\,\mathrm{e}^{\mathrm{i}\Sigma}, \quad (6.5b)$$
$$\Xi_1=u_{0h}(h)\,(1+2\alpha)+\psi_{1hh}, \quad (6.5c)$$

and
$$c_s=c\left(\frac{\Omega}{\kappa}-(1+\alpha)\left(\frac{u_0(0)}{c}-1+2\alpha\right)\right). \quad (6.5d)$$

It should be mentioned that the expression for c_s completely coincides with (4.3) obtained for short waves in water. Its value is determined as a solution of the dispersion equation (4.5). $U(h,Z)$ is the wind flow velocity profile disturbed by the long wave. And since the no-slip condition (5.5) is valid for the long-wave disturbance ψ_1, $U(h=0,Z)=0$, i.e. in (6.4*a*) there is no oscillating part of the short-wave phase velocity, arising due to the Doppler shift in the oscillating velocity field of the long wave, since the disturbance of the air flow velocity is exactly equal to the velocity on the water surface due to no-slip conditions on the air–water boundary. The oscillating part is subtracted from the expression for the relative velocity in (6.4*a*), as already mentioned in the Introduction (see (1.5)).

The first boundary condition for the short waves follows from the general kinematical boundary condition (2.8). Substituting expression (2.9) for $H(\xi,\eta,t)$ and expression (6.1*a*) for ψ into (2.8) yields the following kinematical boundary condition for the short-wave disturbance n^s at the zeroth order in μ and at the first order in ka:

$$\frac{\partial n^s}{\partial\tau}+\left(\frac{u_0-c}{c}(1+2\alpha)+\frac{\psi_{1h}}{Lc}\mathrm{e}^{\mathrm{i}\Sigma}\right)\frac{\partial n^s}{\partial\zeta}+(1+\alpha)\frac{1}{c}\frac{\partial\psi^s}{\partial\zeta}\bigg|_{\eta=0}=0.$$

† More strictly the solution to (6.2*a, b*) should be sought as $\psi^s=\chi(h,Z,\Theta)\,\mathrm{e}^{-\mathrm{i}\Omega\tau+\mathrm{i}\Phi(\zeta,Z)}$ but at zeroth order in μ the dependence on Θ can be omitted.

According to the two-scale method the expression for n^s should be sought in the following form:

$$n^s = n(Z)\,e^{-i\Omega\tau + i\Phi(\xi, Z)}.$$

Taking into account the boundary condition (5.4) for ψ_{1h}, the relation (6.3) between the eikonal Φ and the wavenumber κ, and the expression for c_s gives the following form for the short-wave kinematical boundary condition:

$$c_s n = \chi|_{h=0}. \tag{6.6}$$

A similar procedure for the kinematical boundary condition for the wave disturbances in the water gives

$$c_s n = \phi|_{\theta=0},$$

where $\phi(\Theta, Q, \Sigma, T)$ is the short-wave stream function disturbance in the water. Finally the first boundary condition is

$$\chi|_{h=0} = \phi|_{\theta=0}. \tag{6.7}$$

Equation (6.7) expresses the continuity of the vertical velocity across the water–air surface.

The continuity condition for the horizontal velocity follows from (2.10) taking into account the expressions (4.1) and (6.1) for the disturbances of the stream function in the air and in the water. At the first order in ka it is

$$[U_h(0, Z)\,n + \chi_h(1+\alpha)]\,|_{h=0} = [(1-\alpha)\,\phi_\theta + v_{0\theta}(0)\,n]\,|_{\theta=0}.$$

Taking into account (6.6) and the boundary condition (6.7) gives the secondary boundary condition for χ:

$$\chi_h|_{h=0} = -(U_h(0, Z) - v_{0\theta}(0))\,(1-\alpha)\,\phi|_{\gamma=0}/c_s + (1-2\alpha)\,\phi_{0\theta}|_{\gamma=0}. \tag{6.8}$$

Besides the solution of (6.4) satisfying the boundary conditions (6.7), (6.8) should decrease with distance from the water surface.

There are unknown functions $\mathscr{P}$ and $\mathscr{T}$ (normal and tangential stresses in the air on the water surface) in the dispersion equation (4.5). If the solution of (6.4) with the boundary conditions (6.7) and (6.8) is known, then these functions can be found. Using (6.2*c*) to express the short-wave pressure component by $\chi(h, Z)$ and (2.11) and (2.12) to obtain the short-wave normal and tangential viscous stresses in the linear approximation for the short-wave disturbance yields the following expressions for $\mathscr{P}$ and $\mathscr{T}$ at the zeroth order of μ:

$$\mathscr{P} = \frac{1}{L}\left\{(1+\alpha)\,(c_s\chi_h + \chi U_h) + \frac{\nu_a}{i\kappa L}[\chi_{hhh}(1+3\alpha) - 3\kappa^2(1+\alpha)\,\chi_h + 2\kappa\alpha(\chi_{hh} + \kappa^2\chi)]\right\}, \tag{6.9}$$

$$\mathscr{T} = -\frac{1}{L^2}\left\{(1+2\alpha)\,(\chi_{hh} + (1-2\alpha)\,\kappa^2\chi) - 4\alpha\kappa\chi_h + \frac{\chi}{c_s}\,U(0, \Sigma)_{hh}\right\}. \tag{6.10}$$

so all the terms in the dispersion equation for the short waves (4.5) are determined. Some solutions to this equation are discussed next.

First, consider the solution of (4.5) when there is no viscosity ($\nu_a = 0$), air ($\rho_a/\rho_w = 0$) and drift flow ($v_0 = 0$). Then taking into account (4.6) yields

$$c_s = c_0 = \left\{\left(\frac{g}{\kappa}(1-\alpha) + \frac{T\kappa}{L^2}\right)L\right\}^{1/2}.$$

κ obviously depends on Σ: this dependence can be found by considering the ray equations for the short waves. Taking into consideration that Ω is conserved at the rays gives at the first order of approximation in ka.

$$\kappa = \kappa_0(1+\alpha).$$

This expression is fully within the results of Longuet-Higgins & Stewartson (1960, 1962) (see (1.3)). And in the reference frame moving with the water the short-wave frequency is

$$\omega = \Omega + \kappa(1-\alpha) = \frac{c_s}{c}\kappa = \frac{1}{c}\left\{\left(g\kappa_0 + \frac{T\kappa_0^3}{L^2}(1+3\alpha)\right)L\right\}^{1/2},$$

which is within the results of Longuet-Higgins & Stewartson (1960, 1962) as well.

Consider now under which conditions the drift flow in the water can be neglected. For that purpose one can make use of the results of calculations carried out by Valenzuela (1976), from which it follows that if the friction velocity of the wind is smaller than 30 cm s^{-1}, then taking into consideration the wind drift flow in the water yields only a 10–15% increase in the growth rates of the waves with the wavelength greater than 3 cm. Taking this into account allows the drift flow in the water to be neglected, with an accuracy of not less than 10–15%.

In this case the solution of the Rayleigh equation (4.2) is

$$\Psi = A\,\mathrm{e}^{\theta\kappa(1+\alpha)}.$$

And the dispersion relation (4.5), in which function $\mathbb{G}$ is expressed using (6.9) and (6.10) for $\mathscr{P}$ and $\mathscr{T}$, can be written as

$$\frac{c_s^2}{c_0^2} - 1 - \frac{\nu_w}{\mathrm{i}L\kappa}\frac{c_s}{c_0^2}\left(\frac{c_0^2}{c_s^2}(1-2\alpha)+1+2\alpha\right)2\kappa^2$$
$$= \frac{\rho_a}{\rho_w}\frac{c_s}{c_0^2}\left\{(c_s+\chi_h+\chi U_{0h})\frac{1+\alpha}{\kappa} + \frac{\nu_a}{\mathrm{i}\kappa L}\left[\frac{\chi_{hhh}(1+3\alpha)}{\chi\kappa} - \frac{\chi_{hh}}{\chi}\frac{c_0^2}{c_s^2}(1+2\alpha)\right.\right.$$
$$\left.\left. + \kappa\frac{\chi_h}{\chi}\left(-3+\left(-3+4\frac{c_0^2}{c_s^2}\right)\alpha\right) - \kappa^2\frac{c_0^2}{c_s^2} - \frac{\kappa}{c_s}U_{hh}\left(\frac{c_0^2}{c_s^2}+2\alpha\right)\right]\right\}. \quad (6.11)$$

Boundary conditions (6.7), (6.8) are transformed in the following way:

$$\chi(0) = \chi_0, \quad (6.12a)$$

$$\chi_h(0) = \left[\kappa - \frac{U_h(0,\Sigma)}{c_s}\right](1-\alpha)\chi_0. \quad (6.12b)$$

7. The modulation of the growth rate of the short waves in the presence of the long waves

To calculate the modulation of the growth rate of the short waves in the presence of the long waves the dispersion relation (6.11) was considered. The function $\chi(h)$ in (6.11) was determined by numerical calculation of (6.4a, b) with the boundary conditions (6.12) and the condition of decreasing at infinity. Also, (5.11a, b) with the boundary conditions (5.3), (5.5) and the decreasing condition at infinity was solved numerically to find ψ_1. The stretched vertical coordinate y was used instead of h:

$$y = \log(h+10^{-4}).$$

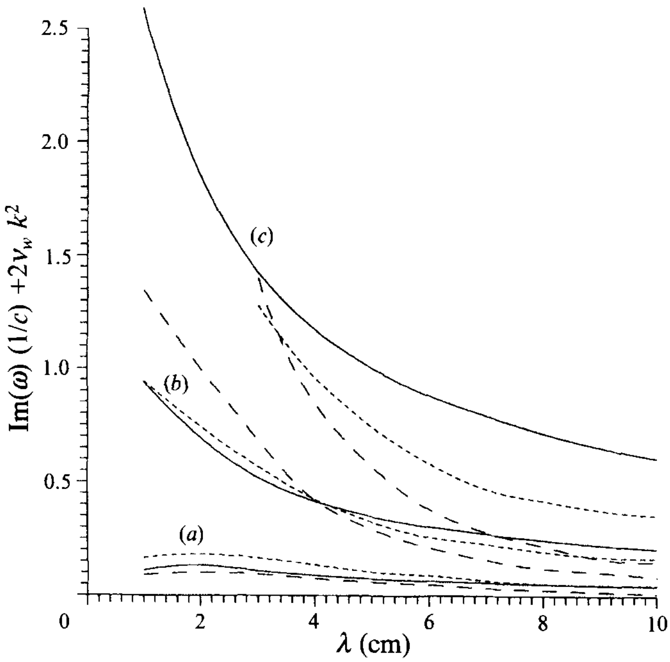

FIGURE 1. Comparison of wind-wave growth rates: ——, calculations using the present model; ------, calculations from the Miles asymptotic theory (1962); ······, calculations by Valenzuela (1976). (*a*) $u_* = 10$ cm s^{-1}, (*b*) $u_* = 20$ cm s^{-1}, (*c*) $u_* = 30$ cm s^{-1}.

A finite-difference approximation, with a constant step in y, of the equations and corresponding boundary conditions leads to a system of linear algebraic equations with pentadiagonal band matrices. The Gauss elimination method modified for a band matrix was used for solving of the systems. Details of the algorithm can be found in Forsythe & Moler (1967).

To verify the validity of the numerical algorithm the imaginary part of c_s was calculated for the case when the air flow velocity was zero and the expression for c_s could easily be found analytically. The error was less than 1%.

To calculate the growth rate of the surface waves β the model logarithmic-linear wind velocity profile was used:

$$\frac{u_0(h)}{10u_*} = U_0(h) = \begin{cases} h, & h < q/10 \\ \dfrac{q}{10} + \dfrac{1}{4}(\alpha - th\alpha), & h > q/10, \end{cases}$$

where $\sinh(\alpha) = 0.8\,(10h - q)$, and q is a constant defining the position of the viscous sublayer. As in Miles (1962) $q = 8$ was taken for atmospheric boundary-layer flow.

To calculate the growth rate β, (6.11) has been solved by the iteration method. First, calculations of the growth rate of short wind waves without a long wave ($\alpha = 0$) were carried out for a number of values of the wind friction velocity $u_* = 10$, 20 and 30 cm s^{-1}. They were compared with the Miles (1962) asymptotic theory and the calculations of Valenzuela (1976). The results are presented on figure 1. All the calculations are in good agreement for $u_* = 10$ and 20 cm s^{-1}. For $u_* = 30$ cm s^{-1} our calculations give values of the growth rate larger than obtained by Miles (1962) and Valenzuela (1976). This difference from the Miles calculations may be caused by the fact that his asymptotic theory is not valid to $u_* = 30$ cm s^{-1} and $\lambda > 3$ cm, since the thickness of

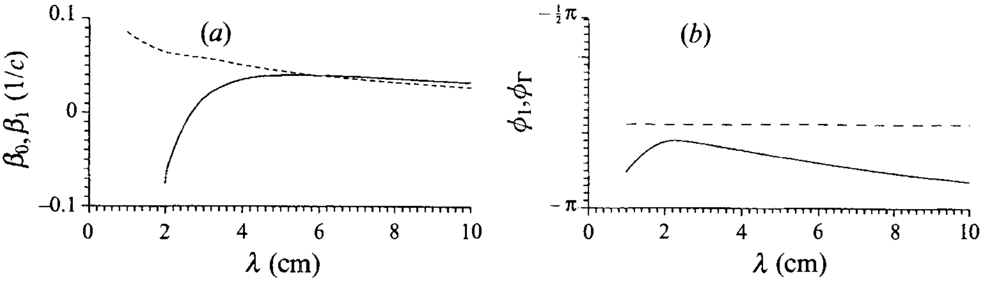

FIGURE 2. The mean growth rate of the short waves and its first harmonics (*a*) and phase (*b*) versus the wavelength of the short wave λ. The parameters of the long wave are: $k = 0.1\ \mathrm{m}^{-1}$, $ka = 0.04$. The wind friction velocity $u_* = 10\ \mathrm{cm\ s}^{-1}$. (*a*) ——, β_0 (the mean growth rate); ------, β_1 (the amplitude of the first harmonic of the growth rate). (*b*) ——, ϕ_1 (the phase of the first harmonic of the growth rate); ----, ϕ_Γ (the phase of the wind velocity gradient near the water surface).

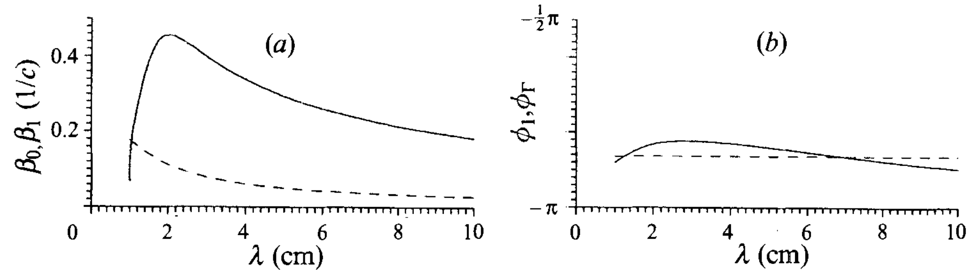

FIGURE 3. As figure 2 but with $u_* = 20\ \mathrm{cm\ s}^{-1}$.

the viscous sublayer of the logarithmic boundary layer (qu_*/ν_a) is equal to or less than the thickness of the viscous sublayer in the wave field $(\nu_a^2/(u_*^2 k))^{3/2}$ (see Valenzuela 1976).

The growth rate of a short wave on a long wave is generally the sum of the constant and periodically varying components, and β can be presented as

$$\beta = \beta_0 + \sum_j \beta_j \cos(\Sigma j - \phi_j). \tag{7.1}$$

For the linear regime of a long-wave disturbance in air only the first harmonic in (7.1) can be correctly considered, since the higher harmonics are of higher order in ka. For the nonlinear regime of a long wave in air the higher harmonics can be correctly taken into account. But, as follows from the numerical calculations, the amplitude of the first harmonic β_1 is essentially larger than the higher ones in all the cases under consideration. The dependencies of β_0, β_1, ϕ_1 on the wavelength of the short wave λ are presented on figures 2, 3 and 4 for the long-wave parameters $k = 10^{-1}\ \mathrm{m}^{-1}$ and $ka = 0.04$. The wind friction velocities are $u_* = 10$, 20 and 30 $\mathrm{cm\ s}^{-1}$ on figures 2, 3 and 4 respectively. The relation of the oscillating component of the wind velocity gradient $\psi_{1\eta\eta}$ to the undisturbed gradient $u_{0\eta}$ has been calculated for each case:

$$M = \frac{\psi_{1\eta\eta}}{u_{0\eta}} = \frac{2kac}{u^2}(ck\nu_a)^{1/2}. \tag{7.2}$$

For $u_* = 10\ \mathrm{cm\ s}^{-1}$, $M = 0.33$ (figure 2), for $u_* = 20\ \mathrm{cm\ s}^{-1}$, $M = 0.082$ (figure 3), for $u_* = 30\ \mathrm{cm\ s}^{-1}$, $M = 0.037$ (figure 4). And when the value of M is of order of unity (the

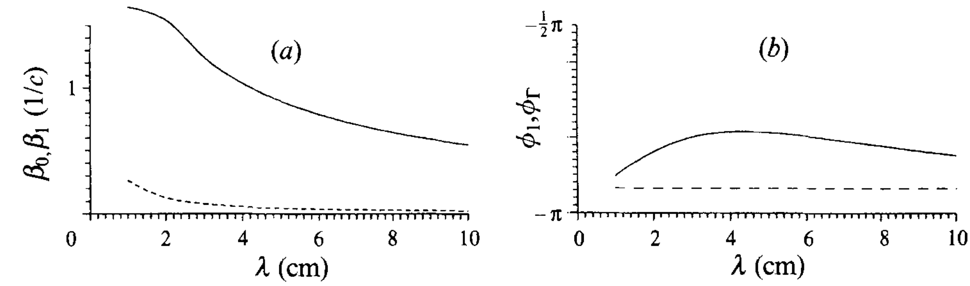

FIGURE 4. As figure 2 but with $u_* = 30$ cm s^{-1}.

nonlinear regime), then the amplitude of the oscillating part of the growth rate β_1 is of order of the mean value β_0 (see figure 2*a*). When M is much smaller than unity (the linear regime) then the amplitude of the oscillating part of the growth rate is much smaller, than the averaged one (see figures 3*a*, 4*a*).

The phase of the oscillating part of the growth rate ϕ_1 is shown by the solid line on the figures 2(*b*), 3(*b*), 4(*b*). The dashed line shows the phase of the oscillating components of the wind velocity gradient near the water surface (ϕ_Γ). ϕ_1 appears to be close to ϕ_Γ, which tends to $-\pi$ with the growth of the wind velocity.

8. Discussion: the effect of the modulation of the growth rate on the modulation of the short-wind-wave spectral components

The effect of the modulation of the growth rate on the modulation of the spectral components of short wind waves is estimated in conclusion. To do that one must substitute the modulating growth rate into the equation for the short-wave spectrum on the long wave (the kinematical equation), calculate the modulation of the spectral components and compare this with the modulation caused only by the transformation of the short-wave spectrum on the variable flow of the long wave. Such a comparison was made, based on the model kinematical equation from Valenzuela & Wright (1979). When the velocity field of the long wave is

$$u = akc \cos kx$$

and the growth rate of the short waves is modulated in the following way:

$$\beta = \beta_0 + \beta_1 \cos(kx - \phi),$$

the modulation of the short-wave spectrum was obtained by Valenzuela & Wright (1979) at the first order of approximation in ka and β_1 as

$$F = F_0(1 + m\cos(kx - \psi)),$$

F, F_0 denoting modulated and undisturbed spectral densities of the short waves. The quantities m and ψ are expressed the following way:

$$m = (a^2 + b^2)^{1/2}; \quad \psi = \arctan(b/a),$$

where

$$a = \frac{\omega^2}{4\beta_0^2 + \omega^2}\left[-\left(\frac{\kappa F_{0\kappa}}{F_0} - \frac{g - \kappa^2 T}{2(g - \kappa^2 T)}\right)ka + \frac{2\beta_1 \sin\phi}{\omega} + \frac{4\beta_0\beta_1\cos\phi}{\omega^2}\right],$$

$$b = \frac{2\omega\beta_0}{4\beta_0^2 + \omega^2}\left[-\left(\frac{\kappa F_{0\kappa}}{F_0} - \frac{g - \kappa^2 T}{2(g - \kappa^2 T)}\right)ka + \frac{2\beta_1 \sin\phi}{\omega} - \frac{\beta_1\cos\phi}{\beta_0}\right].$$

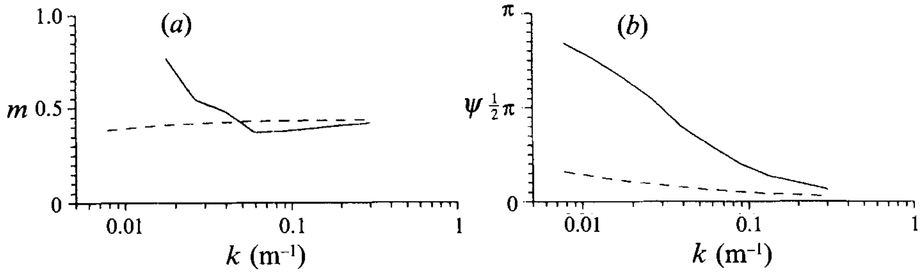

FIGURE 5. The absolute value m (a) and the phase ψ (b) of the modulation coefficient of the spectral component for $\lambda = 5$ cm versus the wavenumber of the long wave k with steepness $ka = 0.1$: ——, taking into account the growth rate modulation; ------, without taking into account the growth rate modulation. Wind friction velocity $u_* = 10\ \mathrm{cm\ s^{-1}}$.

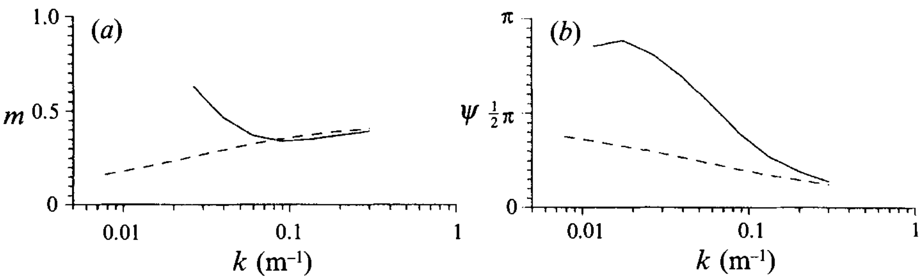

FIGURE 6. As figure 5 but with $u_* = 20\ \mathrm{cm\ s^{-1}}$.

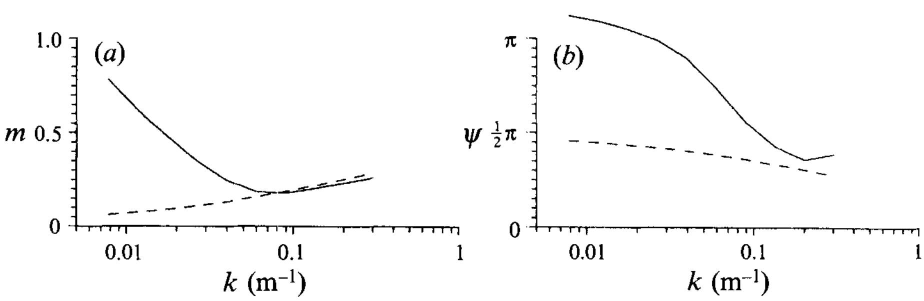

FIGURE 7. As figure 5 but with $u_* = 30\ \mathrm{cm\ s^{-1}}$.

The variation of the absolute value of the modulation coefficient m and the phase ψ of the short-wave spectral component $\lambda = 5$ cm with the wavenumber k of the long wave with steepness $ka = 0.1$ are plotted on figures 5–7. One can see that the modulation of the growth rate of the short wave strongly influences the modulation coefficient of the wave spectrum. For constant steepness of the long wave the effect increases with the growth of the wavelength of the long wave. It is caused, first, by growth of the oscillating component of the air flow velocity gradient like $k^{-1/4}$ and, second, by growth with the long-wave period of the short–long waves interaction time.

The author would like to thank Professor L. A. Ostrovsky for stimulating discussions.

Appendix

The addition to the phase velocity of the short wave caused by the induced flow in the boundary layer on the oscillating surface, which was calculated in Longuet-Higgins (1953), will be estimated here. As for (3.6) one can obviously obtain

$$c_s = c_{s0}\left(1+2k_s\int_{-\infty}^{0}\frac{v_0(\gamma)}{c_s}\mathrm{e}^{2k_s\gamma}\,\mathrm{d}\gamma\right), \tag{A 1}$$

where

$$\frac{\mathrm{d}v_0(\gamma)}{\mathrm{d}\gamma} = 4ck(ka)^2(1-\mathrm{e}^{h/\delta_w})$$

(see Longuet-Higgins 1953).

Integrating (A 1) gives

$$c_s = c_{s0}\left(1+\frac{v_0(0)}{c_{s0}}-\frac{2ck}{k_s c_{s0}}(ka)^2+\frac{c}{c_s}o(\delta k_s)(ka)^2\right).$$

The term $v_0(0)$ is the Doppler shift of the phase velocity, which is not present in the difference $\hat{c}_s - u$ (see (1.5)). The terms $2ck(ka)^2/k_s c_{s0} \ll (ka)^2$, since $\omega = ck \ll \omega_s = c_s k_s$.

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